INTERNATIONAL BACCALAUREATE

MATHEMATICS

Higher Level

Wednesday 10 May 1995 (afternoon)

Paper 1 — 2 hours

This examination paper consists of 20 questions.

The maximum mark for each question is 4.

The maximum mark for this paper is 80.

This examination paper consists of 15 pages.

INSTRUCTIONS TO CANDIDATES

Write your candidate reference number in the box:

DO NOT open this examination paper until instructed to do so.

Answer ALL questions in the spaces provided.

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures as appropriate.

EXAMINATION MATERIALS

Required/Essential:

IB Statistical Tables
Electronic calculator
Ruler and compasses

Allowed/Optional:

A simple translating dictionary for candidates not working in their own language
Millimetre square graph paper

FORMULAE

Trigonometrical identities:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$\text{If } \tan\frac{\theta}{2} = t \text{ then } \sin\theta = \frac{2t}{1+t^2} \text{ and } \cos\theta = \frac{1-t^2}{1+t^2}$$

Integration by parts:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Standard integrals:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c \quad (|x| < a)$$

Statistics: If $(x_1, x_2, \ldots, x_n)$ occur with frequencies $(f_1, f_2, \ldots, f_n)$ then the mean m and standard deviation s are given by

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i} \qquad s = \sqrt{\frac{\Sigma f_i (x_i - m)^2}{\Sigma f_i}}, \qquad i = 1, 2, \ldots, n$$

Binomial distribution:

$$p_x = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

Maximum marks will be given for correct answers. Where an answer is wrong, some marks may be given for a correct method provided this is shown by written working.

1. Given the complex numbers $z_1 = 3 + i$ and $z_2 = 2 - i$, find

(a) $z_1 z_2$;

(b) $\dfrac{z_1}{z_2}$.

Working:

Answers:

2. Calculate x and y exactly, as rational numbers, if

$$5^{3x} \times 25^{y} = \frac{1}{5}$$

and

$$7^{x} \times 49^{2y} = 1.$$

Working:

Answers:

3. Find the value of x such that

$$\begin{vmatrix} x & 6 \\ x & x \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}.$$

Working:

Answer:

4. Find the value of x satisfying

$$\arccos x + \arccos(\sqrt{15}x) = \frac{\pi}{2}.$$

Working:

Answer:

5. Find the possible values of the number c if the line $y = cx$ passes through a point of intersection of the circle

$$x^2 + y^2 = 5$$

and the line

$$y + 7x = 15.$$

Working:

Answers:

6. Given the two sets of equations,

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 - x_3 \\ y_2 &= 2x_1 + x_2 + 5x_3 \\ y_3 &= 4x_1 - 2x_2 + x_3 \end{aligned} \qquad \begin{aligned} x_1 &= 2z_1 + z_2 - 3z_3 \\ x_2 &= 4z_1 - z_2 + z_3 \\ x_3 &= z_1 - 6z_2 + 7z_3 \end{aligned}$$

use matrix methods to obtain three equations that express y_1, y_2 and y_3 directly in terms of z_1, z_2 and z_3.

Working:

Answers:

7. Find $\frac{d^2y}{dx^2}$ if $y = \frac{1}{\sqrt[3]{x^3 + 2}}$.

Working:

Answer:

8. Two dice are rolled. The score is the smaller of the two numbers that appear; if the same number appears on both dice, then the score is that number.

What is the probability that the score is 3?

Working:

Answer:

9. Find, by considering the cube formed by the vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$, the acute angle between two diagonals of a cube.

Working:

Answer:

10. Write down any two conjugate complex numbers z_1 and z_2, neither lying on the real or imaginary axes of the complex plane, that satisfy $z_1 z_2 = 10$.

Working:

Answers:

11. Evaluate $\int \frac{dx}{x(x-1)^2}$.

Working:

Answer:

12. A parallelogram has sides of length 8 cm and 15 cm and the angle θ at one of the vertices is decreasing at the rate of $\frac{\pi}{60}$ radians per minute. What is the rate at which the area of the parallelogram is changing when $\theta = \frac{\pi}{6}$?

Working:

Answer:

13. By first finding an integrating factor, solve the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = x^2.$$

Working:

Answer:

14. The constant terms in the expansions of

$$\left(px^3 + \frac{q}{x^3}\right)^8 \text{ and } \left(px^2 + \frac{q}{x^2}\right)^4$$

are equal, and p and q are both greater than zero. Express p in terms of q.

Working:

Answer:

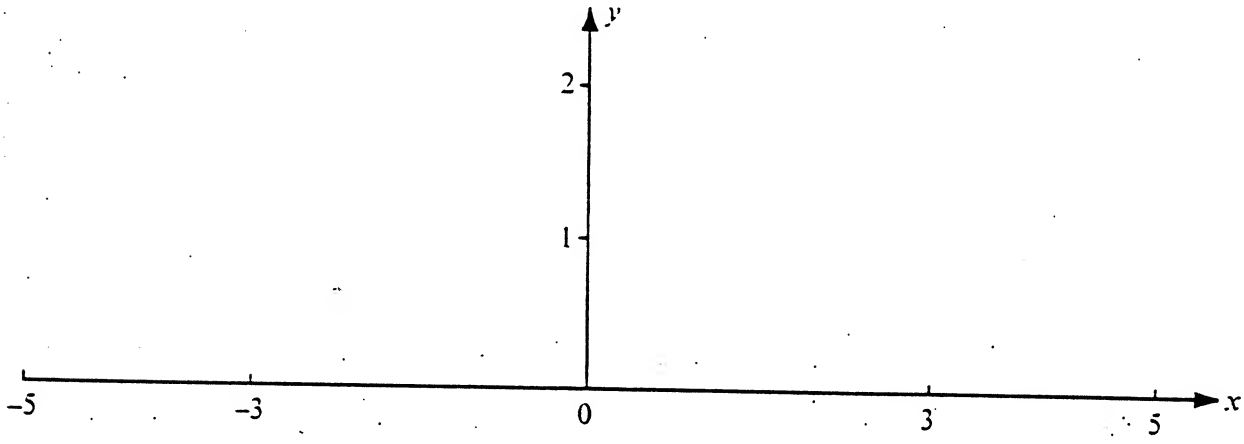
15. Find $|e^{z}|$ if $z = 6\,e^{i\pi/3}$.

Working:

Answer:

16. Sketch on the axes below, the graph of an **even** function $f(x)$ for $-5 < x < 5$ such that

1. $f(0) = 0$ and $f(3) = 1$;
2. $f'(x) > 0$ for $x > 0$; and
3. $f''(x) > 0$ when $0 < x < 3$ and $f''(x) < 0$ when $x > 3$.



Working:

17. A triangle ABC has sides whose lengths are $AB = 5$ cm, $BC = 6$ cm and $CA = 9$ cm. Find the angles of the triangle.

Working:

Answers:

18. Find the values of ε in order that the function $x = a\cos(\omega t + \varepsilon)$ satisfies the differential equation

$$\frac{d^2x}{dt^2} = -\omega^2 x .$$

Working:

Answer:

19. Each **odd** number from 1 to $3n$, where $n \in \mathbb{N}$ and n is odd, is written on a disc and the discs are placed in a box.

(a) How many discs are there in the box?

(b) What is the probability, in terms of n, that a disc drawn at random from the box has a number that is divisible by 3?

Working:

Answers:

20. The graphs given below are those of the same function $f(x)$ for $a \leqslant x \leqslant b$.

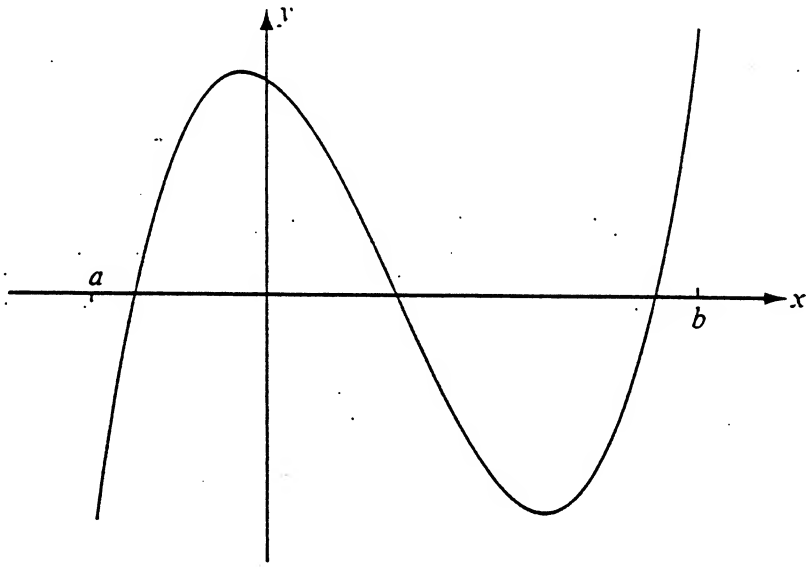
Sketch on the given axes, the graphs of

(a) $|f(x)|$;

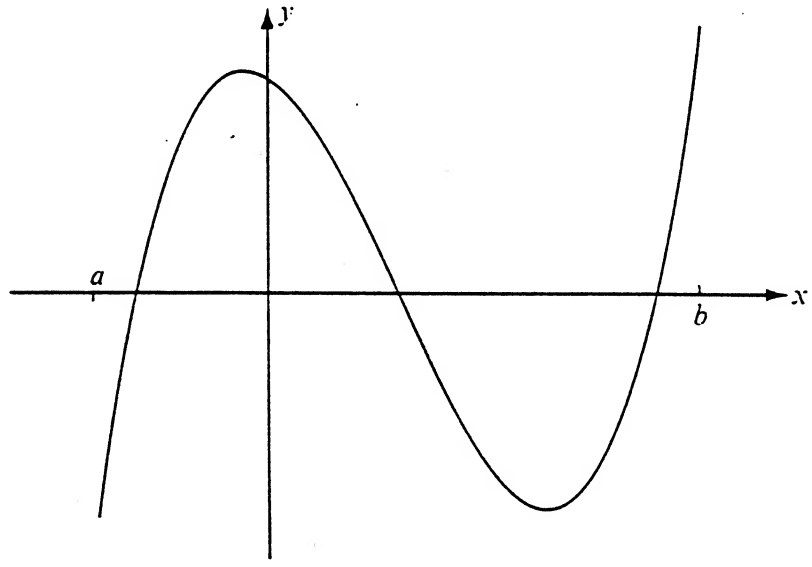
(b) $\dfrac{1}{f(x)}$.

Indicate clearly the positions of any asymptotes.

(a)



(b)



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(*Question 20 continued*)

Working: